
Efficient computation of the cdf of the maximal difference between Brownian bridge and its concave majorant

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Abstract In this paper, we describe two computational methods for calculating the cumulative distribution function and the upper quantiles of the maximal difference between a Brownian bridge and its concave majorant. The first method has two different variants that are both based on a Monte Carlo approach, whereas the second uses the Gaver-Stehfest (GS) algorithm for numerical inversion of Laplace transform. If the former method is straightforward to implement, it is very much outperformed by the GS algorithm, which provides a very accurate approximation of the cumulative distribution as well as its upper quantiles. Our numerical work has a direct application in statistics: the maximal difference between a Brownian bridge and its concave majorant arises in connection with a nonparametric test for monotonicity of a density or regression curve on $[0, 1]$. Our results can be used to construct very accurate rejection region for this test at a given asymptotic level.

Keywords Brownian bridge · Concave majorant · Gaver-Stehfest algorithm · Monotonicity · Monte Carlo

1 Introduction

Consider the regression model $Y_i = f_0(t_i) + \epsilon_i$, where $t_i = i/n$ for $i = 1, \dots, n$, and conditionally on the regressors t_i 's, $\epsilon_1, \dots, \epsilon_n$ are i.i.d. $\sim (0, \sigma_0^2)$ with $0 < \sigma_0 < \infty$. Suppose that we are interested in knowing whether the true regression curve f_0 is nondecreasing on some sub-interval of $[0, 1]$. [6] considered the nonparametric test based on the maximum difference between the cumulative sum diagram of the observations and its concave majorant, multiplied by \sqrt{n} and divided by any consistent estimator of σ_0 .

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The intuition behind this test is as follows: Under the null hypothesis that f is decreasing on $[0, 1]$, the function $\int_0^x f_0(t)dt$, $x \in [0, 1]$, is concave, and hence the cumulative sum diagram of the data must be “very close” to its concave majorant as $n \rightarrow \infty$. [6] showed that asymptotically, the Type I error of the test attains its maximum when f_0 is constant on $[0, 1]$. Then, under this least favorable case, the test statistic converges weakly to the maximum difference between a standard Brownian motion on $[0, 1]$ starting at 0 and its concave majorant. If $(B(t), 0 \leq t \leq 1)$, $B(0) = 0$ denote a standard Brownian motion on $[0, 1]$ starting at 0 and \widehat{B} its concave majorant on $[0, 1]$, then Durot’s test statistic converges weakly to

$$M \stackrel{d}{=} \sup_{t \in [0, 1]} \left(\widehat{B}(t) - B(t) \right). \quad (1)$$

A similar testing problem for densities was considered by [11]. The test can be based on the maximum difference between the empirical distribution and its concave majorant, multiplied by \sqrt{n} . When the true density is uniform on $[0, 1]$, this maximum difference converges weakly to the distribution of M as in the regression setting above.

As proved in Proposition 4 (iii) in [4], one interesting property of the distribution of M is that we can replace B in (1) by a standard Brownian *bridge*; i.e, the distribution of M is also that of the maximum difference between a standard Brownian bridge and its concave majorant. Furthermore, the random variable M can be given under a more useful form. Let M_3 denote the maximum of a Brownian excursion and $(M_{3,1}, M_{3,2}, \dots)$ an infinite sequence of independent random variables distributed as M_3 . If (U_1, U_2, \dots) is an infinite sequence of independent uniform random variables on $[0, 1]$ and (L_1, L_2, \dots) the corresponding uniform stick-breaking process; i.e.,

$$L_1 := U_1, \quad L_2 := (1 - U_1)U_2, \quad L_3 := (1 - U_1)(1 - U_2)U_3, \dots$$

then [4] proved in Theorem 1 that

$$M \stackrel{d}{=} \max_j \sqrt{L_j} M_{3,j}. \quad (2)$$

Absolute continuity of M is an immediate corollary of (2). Two other corollaries will follow from the same equality in distribution giving formulae for F , the cdf of M . If F_3 is the cdf of M_3 , then

$$F(x) = E \left[\prod_j F_3 \left(\frac{x}{\sqrt{L_j}} \right) \right], \quad \forall x > 0 \quad (3)$$

The expectation in the formula above is taken with respect to the stick-breaking process, and F_3 is known to be given by

$$F_3(x) = 1 - 2 \sum_{n=1}^{\infty} (4n^2 x^2 - 1) \exp(-2n^2 x^2), \quad \forall x > 0,$$

see e.g. [10] and [5].

A second formula, which follows from Proposition 7 and Theorem 8 of [4], gives F as a function of the inverse of a Laplace transform. Let K_m denote the modified Bessel function of the second kind and order $m \in \mathbb{N}$, and G the function defined by

$$G(t) = \prod_{j=1}^{\infty} \exp \left(-4 \left(2\sqrt{2}tn K_1(2\sqrt{2}nt) - K_0(2\sqrt{2}nt) \right) \right), \quad t > 0. \quad (4)$$

Then,

$$F(x) = L^{-1} \left(\frac{G(\sqrt{t})}{t} \right) \left(\frac{1}{x^2} \right), \quad x > 0 \quad (5)$$

where $L^{-1}h(z)$ denotes the value of the inverse of Laplace transform of h at z .

We describe in Section 2 and 3 the implementation of two variants based on a Monte Carlo approach and a Gaver-Stehfest algorithm for approximating the inverse of Laplace transform. If the Monte Carlo (MC) methods are easy to implement, they both require a very large number of simulations in order to obtain the same precision as the deterministic Gaver-Stehfest (GS) algorithm. For $x \geq 0.33$, GS is able to approximate very accurately the cumulative distribution of M at x using a multiple precision library. For values x below what it seems to be a cut-off point for both methods, it is difficult to get a precise approximation for the distribution of M . This problem does not affect the calculation of the upper quantiles which is one of the main motivations of the work. Although the MC approach is not as efficient as the GS algorithm, it seemed natural to describe it in the sequel. We only report the numerical results of the GS algorithm, however. Tables 2, 3 and 4 below give approximated values of the distribution function of M on a grid of real numbers x such that $0.33 \leq x \leq 2.54$ with a regular mesh equal to 0.01. The approximation was performed with a precision ensuring up to 60 significant digits. A table of quantiles of order $p \in \{0.90, 0.91, \dots, 0.99\}$ is given as well. This table can be compared to the Monte Carlo approximated quantiles obtained by [6]. All the code used in the numerical computations in this paper is available at http://www.ceremade.dauphine.fr/~fadoua/bf2010_code/.

2 Monte Carlo approach

We consider two different MC-based algorithms. They have the advantage of being very easy to understand and implement. The first approach is straightforwardly based on the expression of the distribution function of M given in (3). Because of the infinite product in (3), a first approximation due to the truncation of the product is introduced. Control of the error due to this approximation is important in order to obtain a good theoretical estimator. Let $J > 0$ be some finite integer and consider the problem of estimating

$$F_J(x) = E \left[\prod_{j=1}^J F_3 \left(\frac{x}{\sqrt{L_j}} \right) \right], \quad \forall x > 0.$$

For $x > 0$ and a given $\epsilon \in (0, 1/4)$, the following lemma gives a lower bound for J so that

$$0 \leq F_J(x) - F(x) < 2\epsilon. \quad (6)$$

Lemma 1 *The approximation error satisfies (6) if*

$$J \geq J_0 = \left\lceil \frac{-\log(x^2 \epsilon^2 / 2)}{\log(2)} \right\rceil + 1. \quad (7)$$

Proof. See Appendix.

For $x > 0$ and a given $\epsilon > 0$, we draw C independent copies $(L_1^{(c)}, L_2^{(c)}, L_3^{(c)}, \dots, L_J^{(c)})$ for $c = 1, \dots, C$ to estimate $F_J(x)$ where $J = J_0$ as given in Lemma 6. The resulting Monte Carlo estimator is

$$\hat{F}_{J,C}(x) = \frac{1}{C} \sum_{c=1}^C \prod_{j=1}^J F_3\left(\frac{x}{\sqrt{L_j^{(c)}}}\right).$$

The computation of the distribution function F_3 imposes yet another approximation due to the fact that it is defined through an infinite series. The number of terms in the approximating finite sum needs to be larger for smaller values of x . Now by the Central Limit Theorem, we have

$$\sqrt{C}(\hat{F}_{J,C}(x) - F_J(x)) \rightarrow_d \mathcal{N}(0, \sigma_J^2)$$

with

$$\sigma_J^2 = \text{Var} \left[\prod_{j=1}^J F_3\left(\frac{x}{\sqrt{L_j}}\right) \right].$$

Let $z_{1-\alpha/2}$ be the $(1 - \alpha/2)$ -quantile of a standard normal for some small $\alpha \in (0, 1)$. Then, for C large enough the event

$$F_J(x) \in \left[\hat{F}_{J,C}(x) - \frac{\sigma_J z_{1-\alpha/2}}{\sqrt{C}}, \hat{F}_{J,C}(x) + \frac{\sigma_J z_{1-\alpha/2}}{\sqrt{C}} \right]$$

occurs with probability $\approx 1 - \alpha$. Combining both the deterministic and Monte Carlo approximations and noting that $\sigma_J^2 \in [0, 1]$, it follows that

$$F(x) \in \left[\hat{F}_{J,C}(x) - 2\epsilon - \frac{z_{1-\alpha/2}}{\sqrt{C}}, \hat{F}_{J,C}(x) + \frac{z_{1-\alpha/2}}{\sqrt{C}} \right]$$

occurs with at least probability $\approx 1 - \alpha$. Hence, to ensure an error of order ϵ , the sample size C should be chosen of order $\lfloor 1/\epsilon^2 \rfloor$. Therefore, very large sample sizes are needed to get accurate results. To give an order of magnitude, Table 1 shows several values of J_0 and C corresponding to desired precision targets. All the values are computed for $0.33 \leq x \leq 2.54$, where 0.33 appears to be the numerical limit of what we can compute without violating the basic properties of a distribution function. This point will be brought up again in the next section. Note that the main purpose of Table 1 is to give an idea about how J_0 and C behave as functions of the precision. For instance, a precision of order 10^{-5} is useless if the goal is to compute an approximation of the value distribution function of M at 0.33 since it is of order 10^{-12} as found with the GS algorithm.

We use the above MC approach to estimate the distribution function of M for $0.33 \leq x \leq 2.54$ as well as the upper quantiles. The algorithm is implemented in C. This method turns out to be very slow for large sample sizes. Moderate sample sizes (of order 10^6) do not give the desired accuracy for small x . The estimates of the distribution function for large x (of order 0.80 and above) as well as the upper quantiles match with those obtained by GS algorithm (see next section).

In the same vein, one can consider a second variant of MC. It is mainly based on the following result due to Kennedy 1976 (see Corollary on page 372):

$$M_3 =_d \sup_{t \in [0,1]} B^{\text{br}}(t) - \inf_{t \in [0,1]} B^{\text{br}}(t)$$

where B^{br} is a Brownian bridge on length 1. Now using the well-known Donsker approximation, the distribution of M_3 can be approximated for large N by the distribution of the random variable

$$V_N = \sup_{t \in [0,1]} \sqrt{N}(\mathbb{G}_N(t) - t) - \inf_{t \in [0,1]} \sqrt{N}(\mathbb{G}_N(t) - t)$$

where \mathbb{G}_N is the uniform empirical process based on N independent uniform random variables U_1, \dots, U_N in $[0, 1]$. Using the fact that \mathbb{G}_N is a constant function between the order statistics $U_{(1)} < \dots < U_{(n)}$, it can be easily shown that

$$V_N = \sqrt{N} \left\{ \max_{1 \leq i \leq N} \left(\frac{i}{N} - U_{(i)} \right) - \min_{1 \leq i \leq N} \left(\frac{i-1}{N} - U_{(i)} \right) \right\}.$$

Now the formula in (2) yields the weak approximation

$$M_{J,N} = \max_{1 \leq i \leq J} \sqrt{L_i} V_N^{(j)}$$

where $V_N^{(1)}, V_N^{(2)}, \dots$ are independent random variables distributed as V_N , and J is a positive integer that should be chosen large enough to have the truncation error under control as done above. The distribution function of M can be estimated empirically by generating C independent random variables $M_{J,N}^{(1)}, M_{J,N}^{(2)}, \dots, M_{J,N}^{(C)}$ with the same distribution as $M_{J,N}$. If this second variant of MC has the drawback of adding another error due to the stochastic approximation of F_3 by that of V_N , it gives the possibility to generate samples with a distribution close to that of M for J, N and C large enough. We will not pursue here the calculation of the approximation error as a function of J, N and C , which have to be very large to achieve high precision.

The plot in Figure 2 shows an estimation of F using the first MC method with $J_0 = 100$ and $C = 10,000$. If the values are not accurate for small x , the plot gives nevertheless a good idea about the true shape of F . This is confirmed by the approximation results we obtain with the numerical inversion of Laplace transform. The trajectory of 1000 independent random variables with the same distribution of $M_{J,N}$ for $J = 100$ and $N = 10,000$ is shown in Figure 3. The sample was extracted from a larger one of size 10,000 with an empirical mean and standard deviation equal to 0.9970 and 0.2475 respectively.

If the MC approach gives a first idea of the support and shape of the distribution of M , it is not satisfactory in terms of efficiency and precision. As we show in the next section, the GS algorithm is a much better choice in both respects.

3 Gaver-Stehfest algorithm

The Gaver-Stehfest (GS) algorithm is one of several algorithms of numerical inversion of Laplace transform. For an excellent description of these algorithms, see [1]. The GS algorithm is different from other inversion procedures in that it involves only real

numbers, but it also requires a very high numerical precision as we explain below (also see [1], p. 415). If g is the Laplace transform of some function f defined on \mathbb{R} , then GS approximation of f is given by

$$\tilde{f}_K(t) = \frac{\ln(2)}{t} \sum_{k=1}^{2K} \xi_k g\left(\frac{k \ln(2)}{t}\right) \quad (8)$$

where K is an integer in \mathbb{N}^* and

$$\xi_k = \frac{(-1)^{k+K}}{K!} \sum_{j=\lfloor (k+1)/2 \rfloor}^{k \wedge K} j^{K+1} \binom{K}{j} \binom{2j}{j} \binom{j}{k-j}, \quad 1 \leq k \leq 2K.$$

Under the assumption that the inverse of Laplace transform f has all its singularity points in $(-\infty, 0]$ and that is infinitely differentiable on $(0, \infty)$, an extensive computation study carried out by [2] has shown that

$$\left| \frac{\tilde{f}_K(t) - f(t)}{f(t)} \right| \approx 10^{-0.8K}, \quad t > 0.$$

If the function f is bounded by 1 say, then the approximation in (8) for well-behaved functions (in the sense given above) coincides with the truth up to significant $0.8K$ digits. Hence, the bigger K is, the better is the approximation. However, for large values of K , the binomial coefficients in ξ_k become extremely large and require high numerical precision. Such a facility is typically provided by a Multiple Precision (MP) numerical library or is built-in in some programming languages.

For a given integer $K > 0$, let \tilde{F}_K denote the GS approximation of F . From the formula of F in (5) and (8), it is easily seen that

$$\tilde{F}_K(x) = \sum_{k=1}^{2K} \frac{\xi_k}{k} G(\sqrt{k \log(2)} x), \quad x > 0 \quad (9)$$

where G is the same function defined by the infinite product in (4).

For $x > 0$ and a given $\epsilon > 0$ we approximate G by the product of the first N terms, where N is a positive integer depending on x and ϵ . Define

$$G_N(t) = \prod_{j=1}^N \exp\left(-4\left(2\sqrt{2}tnK_1(2\sqrt{2}nt) - K_0(2\sqrt{2}nt)\right)\right), \quad t > 0$$

the truncated version of G . This truncation induces an additional error which we need to control. In fact, in computing the Gaver-Stehfest approximation of the distribution function F , we actually replace \tilde{F}_K in (9) by

$$\tilde{F}_{N,K}(x) = \sum_{k=1}^{2K} \frac{\xi_k}{k} G_N(\sqrt{k \log(2)} x), \quad x > 0. \quad (10)$$

The following shows that the error due to replacing $\tilde{F}_K(x)$ by $\tilde{F}_{N,K}(x)$ does not exceed a given threshold $\epsilon > 0$ provided that N is large enough.

Lemma 2 For $\epsilon > 0$, we have $|\tilde{F}_{N,K}(x) - \tilde{F}_K(x)| \leq \epsilon$ if $N \geq N_0$ where

$$N_0 = \left\lceil \frac{1}{\sqrt{2\ln(2)} x} \left\{ \ln \left(\frac{1}{\epsilon(1 - \exp(-\sqrt{2\ln(2)} x))} \right) + (2K + 1) \ln(K) + 3K + 2 \right\} \right\rceil + 1. \quad (11)$$

Proof. See Appendix.

From Lemma 2 it follows that

$$|\tilde{F}_{N,K}(x) - F(x)| \leq \epsilon + |\tilde{F}_K(x) - F(x)|.$$

The second term in the left side is known to be of order $10^{-0.8K}$, and hence the approximation is of the same order if ϵ is chosen to be $o(10^{-0.8K})$, and of order ϵ if the latter dominates and N is chosen to be larger or equal than N_0 given in (11).

We implement the multiple precision calculation of \tilde{F}_K in C++ using two open-source libraries for arbitrary precision computation: the GNU Multiple Precision Arithmetic Library (see [9]) and the Multiple Precision Floating-point Reliable Library (MPFR); see [7]. GMP is an optimized library written in C with assembly code for common inner loops. MPFR is built on top of GMP and adds support for common floating-point operations such as $\exp(x)$.

To approximate the Bessel functions in (4), we use Bessel routines from the ALGLIB library¹ based on piecewise rational and Chebyshev polynomial approximations. We use a precision of 4000 bits to represent multiple precision floating-point numbers. However, the provided ALGLIB Bessel approximations only guaranty a maximal error of order 10^{-14} . As a proof-of-concept, we have also implemented the same algorithm using a much slower but more accurate numerical library in Python². For small values of x such as 0.30, 0.31, and 0.32, and unlike with the C library, we obtain results consistent with the monotonicity and positivity of a cumulative distribution function. For $K = 60$, $N = 3200$, the Python code gives the following approximations $9.8605317729e-14$ for $x = 0.32$, $1.10482969e-12$ for $x = 0.32$ and $9.67030359e-12$ for $x = 0.33$.

Computing $\tilde{F}_K(x)$, $x \in [0.33, 2.54]$ takes about 6 hours (90 seconds per function evaluation) on a 2GHz single-processor machine. The computation is dominated by the evaluation of G in (4). The coefficients ξ_k , $k = 1, \dots, 2K$ need to be computed only once. Tables 2, 3 and 4 give the approximated values of F on a grid starting at 0.33 and ending at 2.54 with a regular mesh chosen to be equal 0.01.

Finally, computing the upper quantiles of order is crucial when using the Kolmogorov type monotonicity test based on the maximal distance between the empirical cumulative sum diagram (resp. the empirical distribution) in the regression estimation setting (resp. the density estimation setting), see [6] and [11]. The GS algorithm can be easily used to approximate the upper quantiles of order $p \in \{0.90, 0.91, \dots, 0.99\}$.

Note that these quantiles are between 1.33 and 1.72 (see Table 3). For each quantile, we used a binary search and stopped when the difference between the GS approximation

¹ Available at <http://www.alglib.net/specialfunctions/bessel.php>

² Available at <http://code.google.com/p/mathpy/>

of F at the point and the targeted probability falls below a given threshold (10^{-7} in the results we report). The results are shown in Table 5. This table is to be compared with the one published by [6] who obtained the quantiles for the same probabilities using a Monte Carlo approach.

In this paper, Monte Carlo and a numerical inversion of the Laplace transform were used to estimate the distribution function and upper quantiles of M , the maximal difference between a Brownian motion on $[0, 1]$ (or a Brownian bridge of length 1) and its concave majorant. This random variable determines the asymptotic critical region of a nonparametric test for monotonicity of a density or regression curve. We find the numerical inversion of Laplace transform, based here on the Gaver-Stehfest algorithm, to be much more accurate and faster than the Monte Carlo method. Numerical inversion of Laplace transform was then very well adapted to this problem. However, it would not have been possible to use such an efficient method if a Laplace transform representation of the distribution of M was not available, see [4].

Finally, we would like to draw the reader's attention to the earlier computational work of [8] on Chernoff's distribution. The latter appears as the limit distribution of the Grenander estimator; that is the Maximum Likelihood estimator of a decreasing density on $(0, \infty)$. In their work, [8] have also used a mathematical characterization of Chernoff's distribution. This allowed for a very efficient and fast approximation procedure which also outperformed Monte Carlo estimation.

Table 1 Order of the lower bound J_0 and sample size C .

Precision	J_0	C
10^{-5}	38	10^{-10}
10^{-8}	57	10^{-16}
10^{-10}	71	10^{-20}
10^{-20}	138	10^{-40}

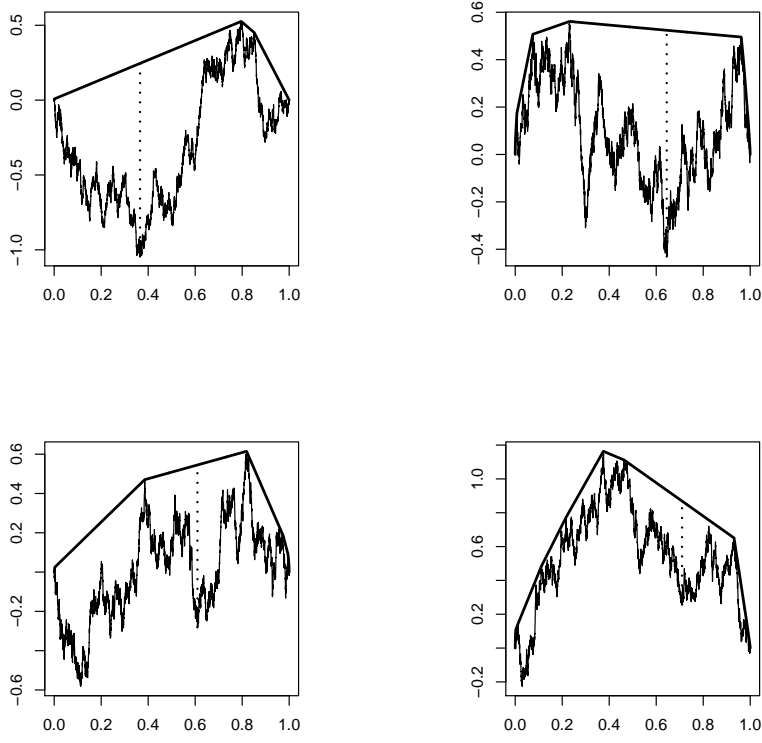


Fig. 1 Four realizations of Brownian bridge and its concave majorant. The length of the dotted vertical segment equals M , the realization of the maximum difference between the Brownian bridge and its concave majorant. The Haar approximation was used to generate the Brownian bridge on a discrete partition of $[0, 1]$ with a mesh equal to 2^{-12} .

Appendix

The following facts will be used in the proof of Lemma 1.

Lemma A.1 *We have*

- (i) For all $j \in \mathbb{N}^*$, $E(L_j) = 1/2^j$.
- (ii) For $x \geq \sqrt{2}$, $F_3(x) \geq \exp(-1/x^2)$.

Proof. The first identity can be proved recursively. For $j = 1$, we have $E(L_1) = E(U_1) = 1/2$. Suppose that $E(L_i) = 1/2^i$ for all $i \leq j$. It is easy to check that

$$L_{j+1} = (1 - U_1)(1 - U_2) \cdots (1 - U_j)U_{j+1} = \left(1 - \sum_{i=1}^j L_i\right)U_{j+1}.$$

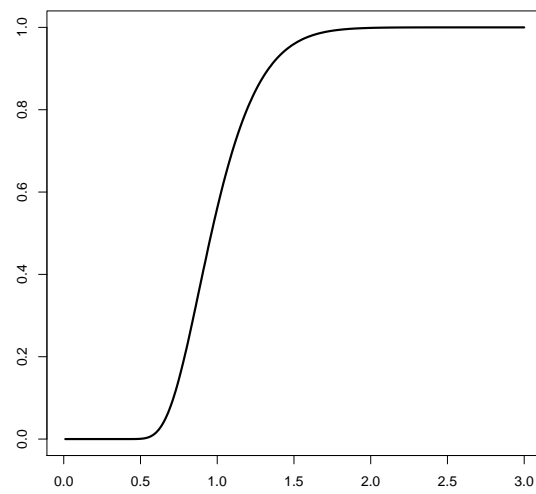


Fig. 2 Plot of a Monte Carlo approximation of F based on a sample of size 10,000, with $J_0 = 100$.

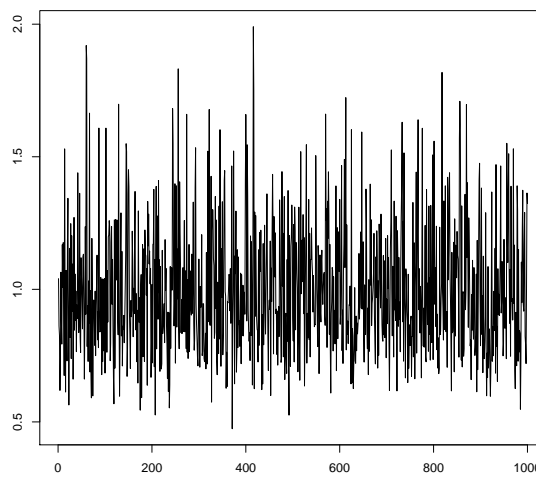


Fig. 3 Plot of the trajectory of random sample of size 1000 of independent realizations of $M_{J,N}$ with $J = 100$ and $N = 10,000$.

Table 2 Approximated values of F obtained by the Gaver-Stehfest algorithm with $K = 100$, $N = 3200$.

x	F_K	x	F_K	x	F_K
0.33	9.24257424322e-12	0.61	1.90415418636e-2	0.89	3.75627377529e-1
0.34	3.74465832649e-10	0.62	2.34012345624e-2	0.90	3.93227821925e-1
0.35	1.96857226601e-9	0.63	2.84117820291e-2	0.91	4.10762393335e-1
0.36	8.39648450672e-9	0.64	3.41073192977e-2	0.92	4.28198510241e-1
0.37	3.13734889037e-8	0.65	4.05156243499e-2	0.93	4.45505815829e-1
0.38	1.04444192578e-7	0.66	4.76576519986e-2	0.94	4.62656184022e-1
0.39	3.13540455103e-7	0.67	5.55472601039e-2	0.95	4.79623704507e-1
0.40	8.57933368778e-7	0.68	6.41911192791e-2	0.96	4.96384649825e-1
0.41	2.16022136991e-6	0.69	7.35887911016e-2	0.97	5.12917427370e-1
0.42	5.04741838646e-6	0.70	8.37329554403e-2	0.98	5.29202518946e-1
0.43	1.10248951989e-5	0.71	9.46097647495e-2	0.99	5.45222410267e-1
0.44	2.26592425775e-5	0.72	1.06199301858e-1	1.00	5.60961512572e-1
0.45	4.40745827182e-5	0.73	1.18476117686e-1	1.01	5.76406078288e-1
0.46	8.15505378035e-5	0.74	1.31409826206e-1	1.02	5.91544112452e-1
0.47	1.44191509163e-4	0.75	1.44965735553e-1	1.03	6.06365281379e-1
0.48	2.44619527193e-4	0.76	1.59105496305e-1	1.04	6.20860819881e-1
0.49	3.99630103610e-4	0.77	1.73787750362e-1	1.05	6.35023438140e-1
0.50	6.30744831947e-4	0.78	1.88968766384e-1	1.06	6.48847229167e-1
0.51	9.64597141746e-4	0.79	2.04603050312e-1	1.07	6.62327577632e-1
0.52	1.43309828337e-3	0.80	2.20643921928e-1	1.08	6.75461070708e-1
0.53	2.07334764606e-3	0.81	2.37044050697e-1	1.09	6.88245411417e-1
0.54	2.92727237058e-3	0.82	2.53755946188e-1	1.10	7.00679334912e-1
0.55	4.04100308041e-3	0.83	2.70732400181e-1	1.11	7.12762527962e-1
0.56	5.46401279238e-3	0.84	2.87926879124e-1	1.12	7.24495551883e-1
0.57	7.24806261908e-3	0.85	3.05293866924e-1	1.13	7.35879769041e-1
0.58	9.44600944518e-3	0.86	3.22789159067e-1	1.14	7.46917273011e-1
0.59	1.21105368429e-2	0.87	3.40370109977e-1	1.15	7.57610822418e-1
0.60	1.52928712783e-2	0.88	3.57995836076e-1	1.16	7.67963778457e-1

By independence of (L_1, \dots, L_j) and U_{j+1} , we can write

$$E(L_{j+1}) = E(1 - \sum_{i=1}^j L_i)/2 = (1 - \sum_{i=1}^j 1/2^i)/2 = 1/2^{j+1}$$

and the identity is proved for all $j \in \mathbb{N}^*$.

For the second inequality, we will use the fact that for a given $a \geq 1$

$$(4a^2t - 1) \exp(-2a^2t) \leq \exp(-at), \quad \text{for all } t \geq 0. \quad (12)$$

Consider the function

$$h(t) := (4a^2t - 1) \exp(-a(2a - 1)t), \quad t \geq 0.$$

The study of variations of h shows that h is increasing on $[0, (6a - 1)/(2a - 1)]$ and decreasing on $[(6a - 1)/(2a - 1), \infty)$ with $h(0) = -1$, $\lim_{t \rightarrow \infty} h(t) = 0$ and

$$h\left(\frac{6a - 1}{2a - 1}\right) = \frac{4a}{2a - 1} \exp\left(-\frac{6a - 1}{4a}\right).$$

Now, the function

$$\log h\left(\frac{6a - 1}{2a - 1}\right) = \log\left(\frac{4a}{2a - 1}\right) - \frac{6a - 1}{4a}$$

Table 3 Approximated values of F obtained by the Gaver-Stehfest algorithm.

x	\tilde{F}_K	x	\tilde{F}_K	x	\tilde{F}_K
1.17	7.77980046011e-1	1.51	9.62020662224e-1	1.85	9.96016083423e-1
1.18	7.87664018322e-1	1.52	9.64212812296e-1	1.86	9.96298535959e-1
1.19	7.97020525083e-1	1.53	9.66292598240e-1	1.87	9.96562371771e-1
1.20	8.06054783852e-1	1.54	9.68264838078e-1	1.88	9.96808708806e-1
1.21	8.14772354647e-1	1.55	9.70134203687e-1	1.89	9.97038605986e-1
1.22	8.23179097587e-1	1.56	9.71905221291e-1	1.90	9.97253065744e-1
1.23	8.31281133430e-1	1.57	9.73582272276e-1	1.91	9.97453036494e-1
1.24	8.39084806872e-1	1.58	9.75169594296e-1	1.92	9.97639415022e-1
1.25	8.46596652448e-1	1.59	9.76671282637e-1	1.93	9.97813048825e-1
1.26	8.53823362907e-1	1.60	9.78091291833e-1	1.94	9.97974738360e-1
1.27	8.60771759907e-1	1.61	9.79433437494e-1	1.95	9.98125239231e-1
1.28	8.67448766898e-1	1.62	9.80701398339e-1	1.96	9.98265264308e-1
1.29	8.73861384073e-1	1.63	9.81898718407e-1	1.97	9.98395485767e-1
1.30	8.80016665251e-1	1.64	9.83028809430e-1	1.98	9.98516537064e-1
1.31	8.85921696569e-1	1.65	9.84094953345e-1	1.99	9.98629014836e-1
1.32	8.91583576893e-1	1.66	9.85100304937e-1	2.00	9.98733480735e-1
1.33	8.97009399815e-1	1.67	9.86047894590e-1	2.01	9.98830463190e-1
1.34	9.02206237159e-1	1.68	9.86940631128e-1	2.02	9.98920459107e-1
1.35	9.07181123890e-1	1.69	9.87781304739e-1	2.03	9.99003935491e-1
1.36	9.11941044337e-1	1.70	9.88572589969e-1	2.04	9.99081331015e-1
1.37	9.16492919660e-1	1.71	9.89317048762e-1	2.05	9.99153057516e-1
1.38	9.20843596472e-1	1.72	9.90017133547e-1	2.06	9.99219501431e-1
1.39	9.24999836546e-1	1.73	9.90675190351e-1	2.07	9.99281025174e-1
1.40	9.28968307546e-1	1.74	9.91293461936e-1	2.08	9.99337968446e-1
1.41	9.32755574715e-1	1.75	9.91874090944e-1	2.09	9.99390649494e-1
1.42	9.36368093452e-1	1.76	9.92419123041e-1	2.10	9.99439366308e-1
1.43	9.39812202742e-1	1.77	9.92930510053e-1	2.11	9.99484397768e-1
1.44	9.43094119365e-1	1.78	9.93410113095e-1	2.12	9.99526004728e-1
1.45	9.46219932852e-1	1.79	9.93859705663e-1	2.13	9.99564431063e-1
1.46	9.49195601129e-1	1.80	9.94280976712e-1	2.14	9.99599904647e-1
1.47	9.52026946811e-1	1.81	9.94675533688e-1	2.15	9.99632638303e-1
1.48	9.54719654107e-1	1.82	9.95044905522e-1	2.16	9.99662830687e-1
1.49	9.57279266289e-1	1.83	9.95390545586e-1	2.17	9.99690667143e-1
1.50	9.59711183695e-1	1.84	9.95713834588e-1	2.18	9.99716320502e-1

is decreasing on $[1, \infty)$ with $\log h(1) = \log(4) - 5/4 < 0$, and hence $h((6a-1)/(2a-1)) <$

1. It follows that $h(t) < 1$ and the inequality in (12) is proved.

It follows that

$$\begin{aligned}
 1 - F_3(x) &= 2 \sum_{k=1}^{\infty} (4k^2 x^2 - 1) \exp(-2k^2 x^2) \leq 2 \sum_{k=1}^{\infty} \exp(-kx^2) \\
 &= \frac{2 \exp(-x^2)}{1 - \exp(-x^2)}.
 \end{aligned}$$

To show that $F_3(x) \geq \exp(-1/x^2)$ for all $x \geq 2$, it is enough to show that

$$\frac{1 - 3 \exp(-x^2)}{1 - \exp(-x^2)} \geq \exp(-1/x^2), \quad \text{for all } x \geq \sqrt{2}$$

or equivalently

$$2 \exp(-t) \leq (1 - \exp(-t))(1 - \exp(-1/t)), \quad \text{for all } t \geq 2.$$

Table 4 Approximated values of F obtained by the Gaver-Stehfest algorithm.

x	$F_K(x)$
2.19	9.99739951848e-1
2.20	9.99761711238e-1
2.21	9.99781738392e-1
2.22	9.99800163331e-1
2.23	9.99817106996e-1
2.24	9.99832681825e-1
2.25	9.99846992298e-1
2.26	9.99860135450e-1
2.27	9.99872201360e-1
2.28	9.99883273608e-1
2.29	9.99893429703e-1
2.30	9.99902741490e-1
2.31	9.99911275534e-1
2.32	9.99919093474e-1
2.33	9.99926252361e-1
2.34	9.99932804973e-1
2.35	9.99938800114e-1
2.36	9.99944282889e-1
2.37	9.99949294965e-1
2.38	9.99953874813e-1
2.39	9.99958057939e-1
2.40	9.99961877094e-1
2.41	9.99965362474e-1
2.42	9.99968541907e-1
2.43	9.99971441026e-1
2.44	9.99974083431e-1
2.45	9.99976490838e-1
2.46	9.99978683223e-1
2.47	9.99980678951e-1
2.48	9.99982494897e-1
2.49	9.99984146562e-1
2.50	9.99985648176e-1
2.51	9.99987012798e-1
2.52	9.99988252405e-1
2.53	9.99989377977e-1
2.54	9.99990399575e-1

Table 5 Approximated upper quantiles $q_{1-\alpha}$ of order $1 - \alpha$. The approximation is based on the Gaver-Stehfest algorithm with $K = 100$, $N = 60$.

α	0.01	0.02	0.03	0.04	0.05
$q_{1-\alpha}$	1.71974853	1.61439819	1.54926391	1.50122253	1.46279052
	0.06	0.07	0.08	0.09	0.10
	1.43055908	1.40267791	1.37802490	1.35586822	1.33570159

The preceding inequality can be proved as follows. Define the function

$$k(t) := (\exp(t) - 1)(1 - \exp(-1/t)), \quad t \geq 2.$$

We will show now that $k(t) \geq 2$ for all $t \geq 2$. For $t \geq 2$, we have

$$k'(t) = \exp(t) \left\{ 1 - \left(1 + \frac{1}{t^2} \right) \exp(-1/t) \right\} + \frac{\exp(-1/t)}{t^2}$$

$$\begin{aligned}
&\geq \exp(t) \left\{ 1 - \left(1 + \frac{1}{t^2} \right) \exp(-1/t) \right\} \\
&= \exp(t) \phi(1/t)
\end{aligned}$$

where

$$\phi(z) = 1 - (1 + z^2) \exp(-z), \quad z \in [0, 1/2].$$

It is easy to show that ϕ is increasing on $[0, 1/2]$ and hence $\phi(z) \geq \phi(0) = 1$. It follows that the function k is increasing on $[2, \infty)$. Since $k(2) \approx 2.514 \geq 0$, the inequality $F_3(x) \geq \exp(-1/x^2)$, $x \geq \sqrt{2}$ follows. \square

Proof of Lemma 1. Define

$$\Delta_J := 1 - \prod_{j=J+1}^{\infty} F_3\left(\frac{x}{\sqrt{L_j}}\right).$$

We have

$$\begin{aligned}
0 \leq F_J(x) - F(x) &= E \left[\prod_{j=1}^J F_3\left(\frac{x}{\sqrt{L_j}}\right) \Delta_J \right] \\
&\leq E[\Delta_J] \\
&= E[\Delta_J 1_{\Delta_J \leq \epsilon}] + E[\Delta_J 1_{\Delta_J > \epsilon}] \\
&\leq \epsilon + P(\Delta_J > \epsilon).
\end{aligned}$$

Let A_J be the event

$$A_J = \left\{ L_j \leq x^2/4, \text{ for all } j \geq J+1 \right\}$$

and A_J^c its complement.

We can write

$$\begin{aligned}
P(\Delta_J > \epsilon) &= P\left(\prod_{j=J+1}^{\infty} F_3\left(\frac{x}{\sqrt{L_j}}\right) < 1 - \epsilon \right) \\
&= P\left(\left\{ \prod_{j=J+1}^{\infty} F_3\left(\frac{x}{\sqrt{L_j}}\right) < 1 - \epsilon \right\} \cap A_J \right) + P(A_J^c) \\
&\leq P\left(\prod_{j=J+1}^{\infty} \exp(-L_j/x^2) < 1 - \epsilon \right) + P(A_J^c), \text{ using Lemma A.1 (i)} \\
&= P\left(\sum_{j=J+1}^{\infty} L_j > x^2 \log(1/(1 - \epsilon)) \right) + P(A_J^c) \\
&\leq P\left(\sum_{j=J+1}^{\infty} L_j > x^2 \log(1/(1 - \epsilon)) \right) + \sum_{j=J+1}^{\infty} P(L_j > x^2/4).
\end{aligned}$$

Using Lemma A.1 (ii) and the Chebyshev inequality, we get

$$\begin{aligned} P\left(\sum_{j=J+1}^{\infty} L_j > x^2 \log(1/(1-\epsilon))\right) &\leq \frac{\sum_{j=J+1}^{\infty} 1/2^j}{x^2 \log(1/(1-\epsilon))} \\ &= \frac{1}{2^J x^2 \log(1/(1-\epsilon))} \end{aligned}$$

and

$$\sum_{j=J+1}^{\infty} P(L_j > x^2/4) \leq \frac{4}{2^J x^2}.$$

Hence,

$$0 \leq F_J(x) - F(x) \leq \epsilon + \frac{1}{2^J x^2 \log(1/(1-\epsilon))} + \frac{4}{2^J x^2}.$$

To have this approximation error smaller than 2ϵ , it suffices to take

$$J > \frac{1}{\log(2)} \left(\log\left(\frac{1}{\epsilon x^2}\right) + \log\left(\frac{1}{\log(1/(1-\epsilon))} + 4\right) \right).$$

If $\epsilon < 1/4$ we can take

$$J \geq \left\lfloor \frac{-\log(x^2 \epsilon^2/2)}{\log(2)} \right\rfloor + 1 \quad (13)$$

and Lemma 6 is proved. \square

Proof of Lemma 2. The modified Bessel function of the second kind K_n is known to converge to 0 as $x \rightarrow \infty$. Moreover we have

$$K_n(x) = \sqrt{\frac{\pi}{2}} \left(\frac{\exp(-x)}{\sqrt{x}} + o\left(\frac{1}{x}\right) \right), \quad x > 0,$$

and

$$xK_1(x) - K_0(x) \leq \sqrt{\frac{\pi}{2}} \sqrt{x} \exp(-x), \quad \forall x > 0$$

see Lemma A.2. For $t > 0$, define

$$H(t) = 4 \sum_{n=1}^{\infty} \left(2\sqrt{2}nt K_1(2\sqrt{2}nt) - K_0(2\sqrt{2}nt) \right)$$

so that $G(t) = \exp(-H(t))$. Also, for $N \in \mathbb{N}^*$ let

$$H_N(t) = 4 \sum_{n=1}^N \left(2\sqrt{2}nt K_1(2\sqrt{2}nt) - K_0(2\sqrt{2}nt) \right)$$

so that $G_N(t) = \exp(-H_N(t))$. We have

$$\begin{aligned}
0 < G_N(t) - G(t) &= \exp(-H_N(t)) - \exp(-H(x)) \leq H(t) - H_N(t) \\
&= 4 \sum_{n=N}^{\infty} (ncK_1(nc) - K_0(nc)) \\
&\leq 4\sqrt{\frac{\pi}{2}} \frac{\exp(-cN/2)}{1 - \exp(-c/2)} \quad (14)
\end{aligned}$$

where $c = 2\sqrt{2}t$.

Let us write again $\tilde{F}_{N,K}$ for the Gaver-Stehfest approximation of the inverse of Laplace transform of G_N .

The corresponding approximation error due to truncating G is given by

$$\tilde{E}_{N,K}(x) = \tilde{F}_K(x) - \tilde{F}_{N,K}(x) = \sum_{k=1}^{2K} \frac{\xi_k}{k} \left(G(\sqrt{k \ln(2)} x) - G_N(\sqrt{k \ln(2)} x) \right), \quad x > 0.$$

By (14), we can write

$$|\tilde{E}_{N,K}(x)| = 4\sqrt{\frac{\pi}{2}} \sum_{k=1}^{2K} \frac{|\xi_k|}{k} \frac{\exp(-\alpha_k N)}{1 - \exp(-\alpha_k)}$$

where $\alpha_k = \sqrt{2 \ln(2) k} x$.

Now, $\exp(-\alpha_k) \leq \exp(-\sqrt{2 \ln(2) x})$ and so $(1 - \exp(-\alpha_k))^{-1} \leq (1 - \exp(-\sqrt{2 \ln(2) x}))^{-1}$ for $k = 1, \dots, 2K$. The coefficients ξ_k can be loosely bounded using the following upper bounds for binomial coefficients

$$\binom{n}{m} \leq \left(\frac{ne}{m}\right)^m, \quad \text{and} \quad \binom{n}{m} \leq \frac{n^m}{m!}.$$

For $k = 1, \dots, 2K$, we have

$$\begin{aligned}
|\xi_k| &\leq \frac{1}{K!} \sum_{j=\lfloor (k+1)/2 \rfloor}^{k \wedge K} j^{K+1} \left(\frac{Ke}{j}\right)^K (2e)^j j^k \\
&\leq \frac{1}{K!} \sum_{j=\lfloor (k+1)/2 \rfloor}^{k \wedge K} k^{k+1} (Ke)^K (2e)^k \\
&\leq \frac{1}{K!} \frac{k}{2} K^{K+2} (2e^2)^K
\end{aligned}$$

so that

$$\sum_{k=1}^{2K} \frac{|\xi_k|}{k} \leq \frac{1}{K!} K^{2K+1} (2e^2)^K.$$

Hence, if we impose that $|\tilde{E}_{K,N}(x)| < \epsilon$, then it is enough to choose N such that

$$N > \frac{1}{\sqrt{2 \ln(2)} x} \left\{ \ln \left(\frac{1}{\epsilon(1 - \exp(-\sqrt{2 \ln(2)} x))} \right) + (2K + 1) \ln(K) + 3K + 2 \right\}. \quad \square$$

Lemma A.2 For all $x > 0$, we have

$$xK_1(x) - K_0(x) \leq \sqrt{\frac{\pi}{2}}\sqrt{x}\exp(-x).$$

Proof. Let us recall some well-known facts about modified Bessel functions of the second kind.

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2}} \frac{\exp(-z)}{\sqrt{z}}, \text{ for all } z \in \mathbb{C}^* \quad (15)$$

$$K_n(z) \approx \sqrt{\frac{\pi}{2}} \frac{\exp(-z)}{\sqrt{z}}, \text{ as } |z| \rightarrow \infty \text{ and } n \in \mathbb{N} \quad (16)$$

$$\lim_{x \searrow 0} K_n(x) = \infty \text{ for all } n \in \mathbb{N} \quad (17)$$

$$K_1(z) \approx \frac{1}{z} \text{ as } |z| \searrow 0 \quad (18)$$

$$(z^n K_n(z))' = -z^n K_{n-1}(z), \text{ for all } z \in \mathbb{C} \text{ and } n \in \mathbb{Z} \quad (19)$$

$$K_n'(z) = -\frac{n}{z} K_n(z) - K_{n+1}(z) \text{ for all } z \in \mathbb{C}^* \text{ and } n \in \mathbb{Z} \quad (20)$$

$$K_\nu(x) \leq K_{\nu'}(x) \text{ for all } x > 0 \text{ and } \nu < \nu' \in \mathbb{R}. \quad (21)$$

see e.g. Abramowitz and Stegun 1964. Note first that by (15), the inequality stated in the lemma is equivalent to

$$xK_1(x) - K_0(x) \leq xK_{1/2}(x), \quad x > 0.$$

From (15), (16) and (17), it follows that

$$\begin{aligned} \lim_{x \searrow 0} (xK_1(x) - K_0(x) - xK_{1/2}(x)) &= -\infty \text{ and} \\ \lim_{x \rightarrow \infty} (xK_1(x) - K_0(x) - xK_{1/2}(x)) &= 0. \end{aligned}$$

Let us write $\psi(x) = xK_1(x) - K_0(x) - xK_{1/2}(x)$, $x > 0$. Suppose now that there exists $x > 0$ such that $\psi(x) > 0$. This would imply that there exists $y > 0$ such that $\psi(y) > 0$ and $\psi'(y) = 0$. Now, using (19) and (20) it follows that

$$\psi'(x) = -xK_0(x) + K_1(x) + \left(x - \frac{1}{2}\right) K_{1/2}(x), \quad x > 0.$$

Hence, y satisfies

$$\begin{aligned} yK_1(y) - K_0(y) &> yK_{1/2}(y) \text{ and} \\ K_1(y) &= \left(\frac{1}{2} - y\right) K_{1/2}(y) + yK_0(y). \end{aligned}$$

It follows that

$$(y^2 - 1)K_0(y) > y\left(y + \frac{1}{2}\right) K_{1/2}(y).$$

Since $K_0(x) > 0$ and $K_{1/2}(x) > 0$ for all $x > 0$, we must have $y > 1$. But if $y > 1$, then the previous inequality implies

$$K_0(y) > \frac{y(y + 1/2)}{y^2 - 1} K_{1/2}(y) > K_{1/2}(y)$$

which is impossible by (21).

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